

Math 1B Midterm 1 Review Answers

[1] Using the “Pythagorean”, reciprocal and quotient identities, $\cosh x = \frac{5}{4}$ and $\operatorname{csch} x = -\frac{4}{3}$.

[2] $\lim_{x \rightarrow \infty} \frac{e^{2x} + 1}{e^{2x} - 1} = \lim_{x \rightarrow \infty} \frac{2e^{2x}}{2e^{2x}} = 1$

[3] $y = \cosh^{-1} x$
 $\Rightarrow \cosh y = x$ and $y \geq 0$
 $\Rightarrow (\sinh y) \frac{dy}{dx} = 1$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y - 1}}$ (NOTE: $y \geq 0 \Rightarrow \sinh y \geq 0$)
 $\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$

[4] $y = \tanh^{-1} x$
 $\Rightarrow \tanh y = x$
 $\Rightarrow \frac{e^{2y} - 1}{e^{2y} + 1} = x$
 $\Rightarrow e^{2y} - 1 = x(e^{2y} + 1)$
 $\Rightarrow e^{2y} - 1 = xe^{2y} + x$
 $\Rightarrow e^{2y} - xe^{2y} = 1 + x$
 $\Rightarrow (1 - x)e^{2y} = 1 + x$
 $\Rightarrow e^{2y} = \frac{1 + x}{1 - x}$
 $\Rightarrow 2y = \ln \frac{1 + x}{1 - x}$
 $\Rightarrow y = \frac{1}{2} \ln \frac{1 + x}{1 - x}$

[5] [a] By Net Change Theorem,

$$s(13) - s(1) = \int_1^{13} s'(t) dt = \int_1^{13} v(t) dt \Rightarrow s(13) = s(1) + \int_1^{13} v(t) dt = 21 + \int_1^{13} v(t) dt \text{ feet}$$

[b] [i] $21 + (12 + 18 + 17) * 4 = 209 \text{ feet}$

[ii] $21 + (14 + 24 + 8) * 4 = 205 \text{ feet}$

[iii] $21 + (18 + 17 + 5) * 4 = 181 \text{ feet}$

[6] $\int_{\pi}^{\frac{3\pi}{2}} 2 \sin x dx = -2$ or $\int_{2\pi}^{3\pi} \sin \frac{1}{2} x dx = -2$

$$\begin{aligned}
[7] \quad & \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \left[\left(-1 + \frac{3i}{n} \right)^2 + 3 \left(-1 + \frac{3i}{n} \right) + 2 \right] \\
&= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left(\frac{3i}{n} + \frac{9i^2}{n^2} \right) \\
&= \lim_{n \rightarrow \infty} \frac{3}{n} \left(\frac{3}{n} \sum_{i=1}^n i + \frac{9}{n^2} \sum_{i=1}^n i^2 \right) \\
&= \lim_{n \rightarrow \infty} \frac{3}{n} \left(\frac{3}{n} \frac{n(n+1)}{2} + \frac{9}{n^2} \frac{n(n+1)(2n+1)}{6} \right) \\
&= \lim_{n \rightarrow \infty} 3 \left(\frac{3(n+1)}{2n} + \frac{3(n+1)(2n+1)}{2n^2} \right) \\
&= 3 \left(\frac{3}{2} + \frac{6}{2} \right) \\
&= \frac{27}{2}
\end{aligned}$$

$$\begin{aligned}
[8] \quad & \int_{-2}^8 x \, dx - \int_{-2}^8 \sqrt{25 - (x-3)^2} \, dx \\
&= \text{area of } 8 \times 8 \text{ triangle} - \text{area of } 2 \times 2 \text{ triangle} - \text{area of semicircle of radius 5 (with center at } (3, 0)) \\
&= \frac{1}{2}(8 \times 8) - \frac{1}{2}(2 \times 2) - \frac{1}{2}\pi(5)^2 = 30 - \frac{25}{2}\pi
\end{aligned}$$

$$\begin{aligned}
[9] \quad & \frac{1}{2} \leq \sin x \leq 1 \text{ on } \left[\frac{\pi}{6}, \frac{\pi}{2} \right] \\
&\Rightarrow \frac{1}{2}x \leq x \sin x \leq x \text{ on } \left[\frac{\pi}{6}, \frac{\pi}{2} \right] \\
&\Rightarrow \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}x \, dx \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \, dx \\
&\Rightarrow \frac{1}{4}x^2 \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \leq \frac{1}{2}x^2 \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&\Rightarrow \frac{1}{4} \left[\left(\frac{\pi}{2} \right)^2 - \left(\frac{\pi}{6} \right)^2 \right] \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \leq \frac{1}{2} \left[\left(\frac{\pi}{2} \right)^2 - \left(\frac{\pi}{6} \right)^2 \right] \\
&\Rightarrow \frac{1}{4} \left(\frac{\pi}{6} \right)^2 (3^2 - 1^2) \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \leq \frac{1}{2} \left(\frac{\pi}{6} \right)^2 (3^2 - 1^2) \\
&\Rightarrow \frac{1}{4} \frac{\pi^2}{36} (8) \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \leq \frac{1}{2} \frac{\pi^2}{36} (8) \quad \Rightarrow \frac{\pi^2}{18} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \leq \frac{\pi^2}{9}
\end{aligned}$$

$$[10] \quad \int_7^{-5} f(x) dx + \int_{-1}^{10} f(x) dx + \int_{10}^7 f(x) dx = \int_7^{-5} f(x) dx + \int_{-1}^7 f(x) dx = \int_{-1}^{-5} f(x) dx$$

$$[11] \quad [a] \quad g(3) = \int_{-2}^3 f(t) dt = \frac{1}{2}(3 \times 3) - \frac{1}{2}(2 \times 1) = \frac{7}{2} \text{ and } g'(3) = f(3) = -1$$

[b] Critical numbers of g occur where $g'(x) = 0$ or is undefined in the domain of g .

By FTC Part 1, since f is continuous on $[-5, 5]$, g is differentiable on $(-5, 5)$, and continuous on $[-5, 5]$.

So the domain of g includes $[-5, 5]$.

So, the critical numbers of g occur where $g'(x) = f(x) = 0$ or is undefined in $[-5, 5]$ ie. at $x = -4, -2, 1$ and 4 .

[c] g is decreasing where $g'(x) < 0$, so g is decreasing on $[-\infty, -4]$ and $[1, 4]$.

[d] Local minima of g occur at critical numbers of g where $g'(x)$ changes from negative to positive, so the local minima of g are $x = -4$ and 4 .

[e] g is concave up where $g'(x)$ is increasing, so g is concave up on $[-\infty, -3]$, $[-2, 0]$ and $[3, \infty]$.

[f] Inflection points of g occur where g is continuous and $g'(x)$ changes from increasing to decreasing, or vice versa, so the inflection points of g are $x = -3, -2, 0$ and 3 .

$$[12] \quad g'(x) = 3x^2 \ln(1+x^6) - 2x \ln(1+x^4)$$

$$g''(x) = 6x \ln(1+x^6) + 3x^2 \frac{6x^5}{1+x^6} - \left(2 \ln(1+x^4) + 2x \frac{4x^3}{1+x^4} \right)$$

$$g''(1) = 6 \ln 2 + 3 \left(\frac{6}{2} \right) - 2 \ln 2 - 2 \left(\frac{4}{2} \right) = 4 \ln 2 + 5$$

$$[13] \quad \frac{d}{dx} \left(4 + \int_a^x \frac{1}{f(t)} dt \right) = \frac{d}{dx} 2\sqrt{x} \qquad 4 + \int_a^x \frac{1}{\sqrt{t}} dt = 2\sqrt{x}$$

$$\Rightarrow \frac{1}{f(x)} = \frac{1}{\sqrt{x}} \qquad \Rightarrow 4 + 2\sqrt{t} \Big|_a^x = 2\sqrt{x}$$

$$\Rightarrow f(x) = \sqrt{x} \qquad \Rightarrow 4 + 2\sqrt{x} - 2\sqrt{a} = 2\sqrt{x}$$

$$\qquad \qquad \qquad \Rightarrow 4 - 2\sqrt{a} = 0$$

$$\qquad \qquad \qquad \Rightarrow a = 4$$

[14] the number of pounds Morgan gained from when he was 8 years old to when he was 15 years old

$$[15] \quad [a] \quad v(t) - v(0) = \int_0^t v'(x) dx = \int_0^t a(x) dx \Rightarrow v(t) = v(0) + \int_0^t a(x) dx = 4 + (3x - x^2) \Big|_0^t = 4 + 3t - t^2$$

$$\int_1^6 v(t) dt = \left(4t + \frac{3}{2}t^2 - \frac{1}{3}t^3 \right) \Big|_1^6 = 4(6-1) + \frac{3}{2}(6^2-1^2) - \frac{1}{3}(6^3-1^3) = 20 + \frac{105}{2} - \frac{215}{3}$$

$$= 20 + 52\frac{1}{2} - 71\frac{2}{3} = \frac{5}{6} \text{ meters}$$

$$[b] \quad v(t) = 4 + 3t - t^2 = (4-t)(1+t) \geq 0 \text{ only on } [-1, 4]$$

$$\begin{aligned} \int_1^6 |v(t)| dt &= \int_1^4 (4+3t-t^2) dt + \int_4^6 -(4+3t-t^2) dt = \left(4t + \frac{3}{2}t^2 - \frac{1}{3}t^3\right)\bigg|_1^4 - \left(4t + \frac{3}{2}t^2 - \frac{1}{3}t^3\right)\bigg|_4^6 \\ &= 4(4-1) + \frac{3}{2}(4^2-1^2) - \frac{1}{3}(4^3-1^3) - \left(4(6-4) + \frac{3}{2}(6^2-4^2) - \frac{1}{3}(6^3-4^3)\right) \\ &= 12 + \frac{45}{2} - 21 - \left(8 + 30 - \frac{152}{3}\right) = 12 + 22\frac{1}{2} - 21 - 8 - 30 + 50\frac{2}{3} = 26\frac{1}{6} \text{ meters} \end{aligned}$$

$$[16] \quad \int_{-1}^2 (6 + 4f(2t+1)) dt = \int_{-1}^2 6 dt + 4 \int_{-1}^2 f(2t+1) dt$$

$$\text{Let } u = 2t+1$$

$$\frac{du}{dt} = 2 \Rightarrow dt = \frac{1}{2} du$$

$$t = -1 \Rightarrow u = -1$$

$$t = 2 \Rightarrow u = 5$$

$$\int_{-1}^2 f(2t+1) dt = \int_{-1}^5 \frac{1}{2} f(u) du = \frac{1}{2} \int_{-1}^5 f(u) du = \frac{7}{2}$$

$$\int_{-1}^2 6 dt + 4 \int_{-1}^2 f(2t+1) dt = 6(2 - (-1)) + 4\left(\frac{7}{2}\right) = 32$$

$$[17] \quad \int_{-6}^6 (x^3 \sqrt{\cosh x} - \sqrt{144 - (x+6)^2}) dx = \int_{-6}^6 x^3 \sqrt{\cosh x} dx - \int_{-6}^6 \sqrt{144 - (x+6)^2} dx$$

$$(-x)^3 \sqrt{\cosh(-x)} = -x^3 \sqrt{\cosh x} \Rightarrow \text{first integrand is odd and continuous, over a symmetric interval}$$

$$\int_{-6}^6 x^3 \sqrt{\cosh x} dx = 0$$

$$\int_{-6}^6 \sqrt{144 - (x+6)^2} dx = \text{area of quarter circle of radius 12 (with center at } (-6, 0)) = \frac{1}{4} \pi (12)^2 = 36\pi$$

$$\int_{-6}^6 (x^3 \sqrt{\cosh x} - \sqrt{144 - (x+6)^2}) dx = -36\pi$$